



# An explicit formula for the inverse of band triangular Toeplitz matrix<sup>☆</sup>

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## Abstract

In order to estimate the condition number of the preconditioned matrix proposed in [F.R. Lin, W.K. Ching, Inverse Toeplitz preconditioners for Hermitian Toeplitz systems, Numer. Linear Algebra Appl. 12 (2005) 221–229], we study the inverse of band triangular Toeplitz matrix. We derive an explicit formula for the entries of the inverse of band lower triangular Toeplitz matrix by means of divided difference and use the formula to estimate the condition number of the preconditioned matrices. In particular, we prove that the minimal eigenvalue of preconditioned matrix is well separated from the origin.

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**Keywords:** Band triangular Toeplitz matrix; Generating function; Divided difference; Condition number

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## 1. Introduction

Toeplitz matrices are structured matrices with entries satisfying  $[T_n]_{jk} = t_{j-k}$ ,  $j, k = 1, 2, \dots, n$ . Toeplitz systems  $T_n \mathbf{x} = \mathbf{b}$  arise in a variety of applications in mathematics and engineering, e.g., signal and image processing, queueing problems, etc, see for instance [4]. A  $2\pi$ -periodic function  $f$  is called the generating function of  $T_n$  if

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$$t_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) e^{-ik\theta} d\theta, \quad k = 0, \pm 1, \dots, \pm(n-1), \dots,$$

where  $i = \sqrt{-1}$ . In the following, we denote the Toeplitz matrix  $T_n$  generated by  $f(\theta)$  as  $\mathcal{T}_n[f]$  or  $\mathcal{T}_n[f(\theta)]$ . If  $\mathcal{T}_n[f]$  is invertible, we denote its inverse by  $\mathcal{U}_n[f]$ , i.e.,

$$\mathcal{U}_n[f] = [\mathcal{T}_n[f]]^{-1}.$$

We note that if  $f$  is real-valued, then  $\mathcal{T}_n[f]$  is Hermitian, see for instance; moreover, all eigenvalues of  $\mathcal{T}_n[f]$  lie in the interval  $[\text{ess min}_{-\pi \leq \theta \leq \pi} f(\theta), \text{ess max}_{-\pi \leq \theta \leq \pi} f(\theta)]$ , see for instance [4]. In particular, if  $f(\theta) \geq 0$  with essentially finite zeros, then  $\mathcal{T}_n[f]$  is positive definite and ill-conditioned [2].

The preconditioned conjugate gradient (PCG) methods for solving ill-conditioned Toeplitz systems have been studied by a number of researchers, see for instance [2,3,6–11]. Let  $f(\theta)$  be a nonnegative function in  $\mathcal{C}_{2\pi}$  ( $\mathcal{C}_{2\pi}$  denotes the set of all  $2\pi$ -periodic continuous functions) with finite zeros of even orders. Suppose all roots of  $f(\theta)$  in  $[-\pi, \pi)$  are  $\theta_1, \dots, \theta_s$  with orders  $2\mu_1, \dots, 2\mu_s$ , respectively. One can write

$$f(\theta) = h(\theta)w(\theta), \quad -\pi \leq \theta < \pi,$$

where

$$w(\theta) = \prod_{k=1}^s (1 - \cos(\theta - \theta_k))^{\mu_k}$$

and  $h \in \mathcal{C}_{2\pi}$  with  $h(\theta) > 0$  in  $[-\pi, \pi)$ . Chan [2] proposed the band-Toeplitz matrix  $\mathcal{T}_n[w]$  as a preconditioner for  $\mathcal{T}_n[f]$ , i.e., the PCG method are applied to the preconditioned system  $(\mathcal{T}_n[w])^{-1} \mathcal{T}_n[f] \mathbf{x} = (\mathcal{T}_n[w])^{-1} \mathbf{b}$ . Later Chan and Tang [5] introduced a more efficient preconditioner  $\mathcal{T}_n[g]$ , where  $g$  was defined to be the minimizer of the relative error  $\|(f - g)/f\|_{\infty}$  over all trigonometric polynomials  $g$  of degree  $k$ , a fixed positive integer. One drawback of this approach is that the cost of obtaining  $g$  is expensive. In [11], Serra gave a more practical method by setting  $g = wg_l$ , where  $g_l(\theta) = \sum_{j=0}^l d_j \cos j\theta$  is obtained by the best Chebyshev approximation or by interpolation. In [9], Noutsos and Vassalos proposed approximating  $f/w$  by rational trigonometric functions.

Recently, we proposed a preconditioner which consists of band triangular Toeplitz matrices and another matrix which is efficient for well-conditioned Toeplitz matrix [8]. Let

$$p(\theta) = \prod_{i=1}^s (1 - e^{i(\theta - \theta_i)})^{\mu_i}, \quad -\pi \leq \theta < \pi. \quad (1.1)$$

The key point is that the difference of  $(\mathcal{T}_n[p])^{-1} \mathcal{T}_n[f](\mathcal{T}_n[\bar{p}])^{-1}$  and the well-conditioned Toeplitz matrix  $\mathcal{T}_n[f/(p\bar{p})]$  is a low rank matrix [8, Lemma 3]:

$$\text{rank}((\mathcal{T}_n[p])^{-1} \mathcal{T}_n[f](\mathcal{T}_n[\bar{p}])^{-1} - \mathcal{T}_n[f/(p\bar{p})]) \leq 2\mu,$$

where  $\mu = \sum_{i=1}^s \mu_i$  and  $\bar{z}$  denotes the conjugate of  $z$ . Notice that  $\mathcal{T}_n[p]$  is a band lower triangular Toeplitz matrix and  $\mathcal{T}_n[\bar{p}] = (\mathcal{T}_n[p])^*$ . By using  $p(\theta)\bar{p}(\theta) = 2^\mu w(\theta)$  and  $(\mathcal{T}_n[p])^{-1} = \mathcal{U}_n[p]$ , we can rewrite the above result as

$$\text{rank}(\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}] - \mathcal{T}_n[f/(2^\mu w)]) \leq 2\mu.$$

Therefore, any efficient preconditioner of  $\mathcal{T}_n[f/(2^\mu w)]$  is also efficient for the system

$$\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}] (\mathcal{T}_n[\bar{p}] \mathbf{x}) = \mathcal{U}_n[p] \mathbf{b}.$$

**Theorem 1** [8]. Let  $Q_n$  be an  $n \times n$  positive definite matrix such that

$$Q_n \mathcal{T}_n[f/(2^\mu w)] = I_n + S_n + L_n \quad (1.2)$$

with  $\|S_n\|_2 \leq \varepsilon$  and  $\text{rank}(L_n) \leq v$ , where  $I_n$  is the identity matrix. Then we have

$$Q_n \{\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}]\} = I_n + S_n + \tilde{L}_n,$$

where  $\text{rank}(\tilde{L}_n) \leq 2\mu + v$ .

In [8], we did not study the condition number and the minimal eigenvalue of  $\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}]$  theoretically. We showed numerically that for  $w(\theta) = (1 - \cos(\theta))^l$  with  $l = 1, 2, 3$ , i.e.,  $p(\theta) = (1 - e^{i\theta})^l$ , the condition number and the minimal eigenvalue of  $\mathcal{U}_n[p] \mathcal{T}_n[w] \mathcal{U}_n[\bar{p}]$  are  $O(n^{2l-1})$  and  $2^{-l+1}$  respectively. Based on the numerical tests, we conjectured that the minimal eigenvalue of  $\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}]$  is well separated from the origin. In Section 4, we will prove the conjecture, and study the condition number of  $\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}]$  by investigating the entries of  $\mathcal{U}_n[p]$  which can be represented explicitly by using divided difference with repeated nodes.

The outline of the paper is as follows. In Section 2, we recall the definition of divided difference and give some basic properties of divided difference which are useful in deducing the formula for the entries of  $\mathcal{U}_n[p]$ . In Section 3, we give an explicit formula for the entries of  $\mathcal{U}_n[p]$  by means of divided differences with repeated nodes. In Section 4, we prove that the minimal eigenvalue of  $\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}]$  is well separated from the origin and based on the explicit formula, we show that the condition number of  $\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}]$  is bounded by  $O(n^{2\mu_{\max}-1})$ , where  $\mu_{\max} = \max\{\mu_1, \dots, \mu_s\}$ .

## 2. Divided difference

Divided difference is an elementary concept in numerical analysis, which is originally defined for functions with real variable [1]. Divided difference can be used to represent the Newton's interpolatory formula and to solve Vandermonde systems. We can extend it to functions with complex variable straightforward. In the section, we consider divided difference with repeated nodes for polynomials defined in the complex plane  $\mathbb{C}$ , with which we will derive an explicit formula for the entries of  $\mathcal{U}_n[p]$  in Section 3, where  $p$  is defined by (3.1) ( $p$  defined by (1.1) is a special case). We believe that this topic must have been studied by other researchers, however, we cannot find any relevant reference (we only found a reference in Chinese [13], which considered the divided difference with repeated nodes for functions with real variable).

We first recall the definition of divided difference with different nodes (without repeating). Let  $f(z)$  be a function defined in the complex plane  $\mathbb{C}$ , the first order divided difference is defined as

$$f[z_1, z_2] = \frac{f(z_1) - f(z_2)}{z_1 - z_2}, \quad z_1, z_2 \in \mathbb{C}, \quad z_1 \neq z_2$$

and the divided difference of order  $k$  is defined recursively as

$$f[z_1, z_2, \dots, z_k, z_{k+1}] = \frac{f[z_1, \dots, z_k] - f[z_2, \dots, z_{k+1}]}{z_1 - z_{k+1}},$$

where  $z_1, \dots, z_{k+1} \in \mathbb{C}$  are different pairwise.

It is well-known that

$$f[z_1, z_2, \dots, z_{k-1}, z_k] = \sum_{j=1}^k \frac{f(z_j)}{\prod_{r=1, r \neq j}^k (z_j - z_r)} \quad (2.1)$$

$$= \frac{\det(B)}{\det(A)}, \quad (2.2)$$

where  $\det(\cdot)$  denotes the determinant of a square matrix and

$$A = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{k-1} \\ 1 & z_2 & z_2^2 & \cdots & z_2^{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & z_k & z_k^2 & \cdots & z_k^{k-1} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & z_1 & z_1^2 & \cdots & z_1^{k-2} & f(z_1) \\ 1 & z_2 & z_2^2 & \cdots & z_2^{k-2} & f(z_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & z_k & z_k^2 & \cdots & z_k^{k-2} & f(z_k) \end{bmatrix}.$$

Notice that  $A$  is the Vandermonde matrix and

$$\det(A) = \prod_{1 \leq i < j \leq k} (z_j - z_i). \quad (2.3)$$

Now we begin the discussion of divided difference with repeated nodes. Suppose  $f(z)$  is a polynomial of degree  $m$  and  $z_1, \dots, z_\mu \in \mathbb{C}$  are different pairwise. Since  $z = z_1$  is a root of  $f(z) - f(z_1)$ , i.e.,  $z - z_1$  is a factor of  $f(z) - f(z_1)$ , we see that

$$f[z, z_1] = (f(z) - f(z_1))/(z - z_1)$$

is a polynomial of  $z$  (and  $z_1$ ) of degree  $m - 1$ . Similarly,  $f[z, z_1, \dots, z_\mu]$  is a polynomial of  $z$  (and  $z_1, \dots, z_\mu$ ) of degree  $m - \mu$ . In particular,  $f[z, z_1, \dots, z_m]$  is a constant and  $f[z, z_1, \dots, z_\mu] \equiv 0$  for  $\mu > m$ . Therefore, the limit

$$\lim_{z_1 \rightarrow z, \dots, z_\mu \rightarrow z} f[z, z_1, \dots, z_\mu]$$

exists and we can define  $f[\underbrace{z, \dots, z}_{\mu+1}]$  as

$$f[\underbrace{z, \dots, z}_{\mu+1}] := \lim_{z_1 \rightarrow z, \dots, z_\mu \rightarrow z} f[z, z_1, \dots, z_\mu].$$

For the sake of simplicity, we use the following notations: let  $[z_1, z_2, \dots, z_\mu]$  be denoted by  $[z_{1:\mu}]$ ,  $[z_{i,1}, z_{i,2}, \dots, z_{i,\mu}]$  be denoted by  $[z_{i,1:\mu}]$ , and  $[\underbrace{z_i, \dots, z_i}_\mu]$  be denoted by  $[z_i^{(\mu)}]$ , respectively.

By using these notations, we define divided difference with repeated nodes as

$$\begin{aligned} f[\underbrace{z_1, \dots, z_1}_{\mu_1}, \underbrace{z_2, \dots, z_2}_{\mu_2}, \dots, \underbrace{z_s, \dots, z_s}_{\mu_s}] &= f[z_1^{(\mu_1)}, \dots, z_s^{(\mu_s)}] \\ &:= \lim_{z_{i,1:\mu_i} \rightarrow z_i^{(\mu_i)} (i=1, \dots, s)} f[z_{1,1:\mu_1}, \dots, z_{s,1:\mu_s}]. \end{aligned} \quad (2.4)$$

We will deduce a formula which is similar to (2.2) for the above divided difference. We first give a result for the following special case.

**Lemma 2.** Let  $f(z)$  be a polynomial of degree  $m$ . Then the divided difference of order  $\mu$  with repeated nodes  $z_1, \dots, z_1$  is given by

$$f[z_1^{(\mu+1)}] = \frac{f^{(\mu)}(z_1)}{\mu!} \quad (2.5)$$

and

$$\begin{aligned} f[z_1^{(\mu)}, z] &= \frac{f^{(\mu)}(z_1)}{\mu!} + \frac{f^{(\mu+1)}(z_1)}{(\mu+1)!}(z - z_1) \\ &\quad + \cdots + \frac{f^{(m)}(z_1)}{m!}(z - z_1)^{m-\mu}, \quad \mu \leq m. \end{aligned} \quad (2.6)$$

**Proof.** We prove the theorem by induction on  $\mu$ . By Taylor expansion

$$f(z) = f(z_1) + f'(z_1)(z - z_1) + \frac{f''(z_1)}{2!}(z - z_1)^2 + \cdots + \frac{f^{(m)}(z_1)}{m!}(z - z_1)^m.$$

It follows that

$$f[z_1, z] = \frac{f(z) - f(z_1)}{z - z_1} = f'(z_1) + \frac{f''(z_1)}{2!}(z - z_1) + \cdots + \frac{f^{(m)}(z_1)}{m!}(z - z_1)^{m-1}$$

and therefore

$$f[z_1, z_1] = \lim_{z \rightarrow z_1} f[z_1, z] = f'(z_1).$$

Hence the formulas (2.5) and (2.6) are correct for  $\mu = 1$ . Assume (2.5) and (2.6) hold for  $\mu = k$ , we show the formulas also hold for  $\mu = k + 1$  in the following. By using (2.5) and (2.6) for  $\mu = k$ , we get

$$\begin{aligned} f[z_1^{(k+1)}, z] &= \frac{f[z_1^{(k)}, z] - f[z_1^{(k+1)}]}{z - z_1} \\ &= \frac{\frac{f^{(k)}(z_1)}{k!} + \frac{f^{(k+1)}(z_1)}{(k+1)!}(z - z_1) + \cdots + \frac{f^{(m)}(z_1)}{m!}(z - z_1)^{m-k} - \frac{f^{(k)}(z_1)}{k!}}{z - z_1} \\ &= \frac{f^{(k+1)}(z_1)}{(k+1)!} + \frac{f^{(k+2)}(z_1)}{(k+2)!}(z - z_1) + \cdots + \frac{f^{(m)}(z_1)}{m!}(z - z_1)^{m-k-1}. \end{aligned}$$

Thus (2.6) holds for  $\mu = k + 1$ . Let  $z \rightarrow z_1$ , we get (2.5) for  $\mu = k + 1$ .  $\square$

Let  $q_m(z) = z^m$ , then  $q_m^{(m)}(z) = m!$ . Therefore,

$$q_m[z^{(m+1)}] = q_m^{(m)}(z)/m! = 1, \quad \forall z \in \mathbb{C}.$$

Since  $q_m[z, z_1, \dots, z_m]$  is a polynomial of degree  $m - m = 0$ , i.e.,  $q_m[z, z_1, \dots, z_m]$  is a constant, we have

$$q_m[z_{1:m+1}] = q_m[z_1^{(m+1)}] = 1. \quad (2.7)$$

The following theorem gives a formula for divided difference with repeated notes.

**Theorem 3.** Let  $f(z)$  be a polynomial defined in the complex plane  $\mathbb{C}$  and

$$q_m(z) = z^m.$$

Suppose  $z_1, \dots, z_s \in \mathbb{C}$  are different pairwise,  $\mu_1, \dots, \mu_s$  are positive integers, and  $\mu = \sum_{i=1}^s \mu_i$ . Let

$$A_i = \begin{bmatrix} 1 & q_1(z_i) & \cdots & q_{\mu-1}(z_i) \\ 0 & q'_1(z_i) & \cdots & q'_{\mu-1}(z_i) \\ \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{q_1^{(\mu_i-1)}(z_i)}{(\mu_i-1)!} & \cdots & \frac{q_{\mu-1}^{(\mu_i-1)}(z_i)}{(\mu_i-1)!} \end{bmatrix}_{\mu_i \times \mu} \quad (2.8)$$

and

$$B_i = \begin{bmatrix} 1 & q_1(z_i) & \cdots & q_{\mu-2}(z_i) & f(z_i) \\ 0 & q'_1(z_i) & \cdots & q'_{\mu-2}(z_i) & f'(z_i) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \frac{q_1^{(\mu_i-1)}(z_i)}{(\mu_i-1)!} & \cdots & \frac{q_{\mu-2}^{(\mu_i-1)}(z_i)}{(\mu_i-1)!} & \frac{f^{(\mu_i-1)}(z_i)}{(\mu_i-1)!} \end{bmatrix}_{\mu_i \times \mu} \quad (2.9)$$

for  $i = 1, 2, \dots, s$ . Then we have the following formula for divided difference with repeated nodes:

$$f[z_1^{(\mu_1)}, z_2^{(\mu_2)}, \dots, z_s^{(\mu_s)}] = \frac{\det(B_1^T, \dots, B_s^T)}{\det(A_1^T, \dots, A_s^T)} = \frac{\det(B_1^T, \dots, B_s^T)}{\prod_{1 \leq i < j \leq s} (z_j - z_i)^{\mu_i \mu_j}}.$$

**Proof.** We prove the result by using formulas (2.4), (2.2), and (2.5). Consider the divided difference with different nodes without repeating:

$$f[z_{1,1}, \dots, z_{1,\mu_1}, \dots, z_{s,1}, \dots, z_{s,\mu_s}] = f[z_{1,1;\mu_1}, \dots, z_{s,1;\mu_s}] = \frac{\det(B)}{\det(A)},$$

where

$$A = \begin{bmatrix} 1 & q_1(z_{1,1}) & q_2(z_{1,1}) & \cdots & q_{\mu-1}(z_{1,1}) \\ 1 & q_1(z_{1,2}) & q_2(z_{1,2}) & \cdots & q_{\mu-1}(z_{1,2}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & q_1(z_{1,\mu_1}) & q_2(z_{1,\mu_1}) & \cdots & q_{\mu-1}(z_{1,\mu_1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & q_1(z_{s,1}) & q_2(z_{s,1}) & \cdots & q_{\mu-1}(z_{s,1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & q_1(z_{s,\mu_s}) & q_2(z_{s,\mu_s}) & \cdots & q_{\mu-1}(z_{s,\mu_s}) \end{bmatrix}$$

and

$$B = \begin{bmatrix} 1 & q_1(z_{1,1}) & q_2(z_{1,1}) & \cdots & q_{\mu-2}(z_{1,1}) & f(z_{1,1}) \\ 1 & q_1(z_{1,2}) & q_2(z_{1,2}) & \cdots & q_{\mu-2}(z_{1,2}) & f(z_{1,2}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & q_1(z_{1,\mu_1}) & q_2(z_{1,\mu_1}) & \cdots & q_{\mu-2}(z_{1,\mu_1}) & f(z_{1,\mu_1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & q_1(z_{s,1}) & q_2(z_{s,1}) & \cdots & q_{\mu-2}(z_{s,1}) & f(z_{s,1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & q_1(z_{s,\mu_s}) & q_2(z_{s,\mu_s}) & \cdots & q_{\mu-2}(z_{s,\mu_s}) & f(z_{s,\mu_s}) \end{bmatrix}.$$

By subtracting the first row of  $B$  from the 2nd row,  $\dots$ , and the  $\mu_1$ th row of  $B$ , respectively, we can rewrite the numerator  $\det(B)$  as

$$\det(B) = \det \begin{pmatrix} 1 & q_1(z_{1,1}) & \cdots & q_{\mu-2}(z_{1,1}) & f(z_{1,1}) \\ 0 & q_1(z_{1,2}) - q_1(z_{1,1}) & \cdots & q_{\mu-2}(z_{1,2}) - q_{\mu-2}(z_{1,1}) & f(z_{1,2}) - f(z_{1,1}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & q_1(z_{1,\mu_1}) - q_1(z_{1,1}) & \cdots & q_{\mu-2}(z_{1,\mu_1}) - q_{\mu-2}(z_{1,1}) & f(z_{1,\mu_1}) - f(z_{1,1}) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & q_1(z_{s,1}) & \cdots & q_{\mu-2}(z_{s,1}) & f(z_{s,1}) \\ 1 & q_1(z_{s,2}) & \cdots & q_{\mu-2}(z_{s,2}) & f(z_{s,2}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & q_1(z_{s,\mu_s}) & \cdots & q_{\mu-2}(z_{s,\mu_s}) & f(z_{s,\mu_s}) \end{pmatrix}$$

$$= \prod_{1 < j \leq \mu_1} (z_{1,j} - z_{1,1})$$

$$\times \det \begin{pmatrix} 1 & q_1(z_{1,1}) & \cdots & q_{\mu-2}(z_{1,1}) & f(z_{1,1}) \\ 0 & q_1[z_{1,1}, z_{1,2}] & \cdots & q_{\mu-2}[z_{1,1}, z_{1,2}] & f[z_{1,1}, z_{1,2}] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & q_1[z_{1,1}, z_{1,\mu_1}] & \cdots & q_{\mu-2}[z_{1,1}, z_{1,\mu_1}] & f[z_{1,1}, z_{1,\mu_1}] \\ \vdots & \vdots & & \vdots & \\ 1 & q_1(z_{s,1}) & \cdots & q_{\mu-2}(z_{s,1}) & f(z_{s,1}) \\ 1 & q_1(z_{s,2}) & \cdots & q_{\mu-2}(z_{s,2}) & f(z_{s,2}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & q_1(z_{s,\mu_s}) & \cdots & q_{\mu-2}(z_{s,\mu_s}) & f(z_{s,\mu_s}) \end{pmatrix}.$$

Similarly, subtracting the 2nd row of the above matrix from the 3rd row, ..., and the  $\mu_1$ th row, respectively, and so on, we get

$$\det(B) = \prod_{1 \leq i < j \leq \mu_1} (z_{1,j} - z_{1,i})$$

$$\times \det \begin{pmatrix} 1 & q_1(z_{1,1}) & \cdots & q_{\mu-2}(z_{1,1}) & f(z_{1,1}) \\ 0 & q_1[z_{1,1:2}] & \cdots & q_{\mu-2}[z_{1,1:2}] & f[z_{1,1:2}] \\ \vdots & \vdots & & \vdots & \\ 0 & q_1[z_{1,1:\mu_1}] & \cdots & q_{\mu-2}[z_{1,1:\mu_1}] & f[z_{1,1:\mu_1}] \\ \vdots & \vdots & & \vdots & \\ 1 & q_1(z_{s,1}) & \cdots & q_{\mu-2}(z_{s,1}) & f(z_{s,1}) \\ 1 & q_1(z_{s,2}) & \cdots & q_{\mu-2}(z_{s,2}) & f(z_{s,2}) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & q_1(z_{s,\mu_s}) & \cdots & q_{\mu-2}(z_{s,\mu_s}) & f(z_{s,\mu_s}) \end{pmatrix}$$

$$= \prod_{l=1}^s \left( \prod_{1 \leq i < j \leq \mu_l} (z_{l,j} - z_{l,i}) \right)$$

$$\times \det \begin{pmatrix} 1 & q_1(z_{1,1}) & \cdots & q_{\mu-2}(z_{1,1}) & f(z_{1,1}) \\ 0 & q_1[z_{1,1:2}] & \cdots & q_{\mu-2}[z_{1,1:2}] & f[z_{1,1:2}] \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & q_1[z_{1,1:\mu_1}] & \cdots & q_{\mu-2}[z_{1,1:\mu_1}] & f[z_{1,1:\mu_1}] \\ \vdots & \vdots & & \vdots & \\ 1 & q_1(z_{s,1}) & \cdots & q_{\mu-2}(z_{s,1}) & f(z_{s,1}) \\ 0 & q_1[z_{s,1:2}] & \cdots & q_{\mu-2}[z_{s,1:2}] & f[z_{s,1:2}] \\ \vdots & \vdots & & \vdots & \\ 0 & q_1[z_{s,1:\mu_s}] & \cdots & q_{\mu-2}[z_{s,1:\mu_s}] & f[z_{s,1:\mu_s}] \end{pmatrix}.$$

Notice that if  $f(z) = q_{\mu-1}(z)$ , then  $B = A$ . From (2.3) we see that  $\det(A)$  can be rewritten as

$$\det(A) = \prod_{l=1}^s \left( \prod_{1 \leq i < j \leq \mu_l} (z_{l,j} - z_{l,i}) \right) \times \prod_{\substack{1 \leq k < l \leq s \\ 1 \leq i \leq \mu_l, 1 \leq j \leq \mu_k}} (z_{l,i} - z_{k,j}).$$

From the above formulas for  $\det(A)$  and  $\det(B)$ , it follows that

$$\frac{\det(B)}{\det(A)} = \frac{\det \begin{pmatrix} 1 & q_1(z_{1,1}) & \cdots & q_{\mu-2}(z_{1,1}) & f(z_{1,1}) \\ 0 & q_1[z_{1,1:2}] & \cdots & q_{\mu-2}[z_{1,1:2}] & f[z_{1,1:2}] \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & q_1[z_{1,1:\mu_1}] & \cdots & q_{\mu-2}[z_{1,1:\mu_1}] & f[z_{1,1:\mu_1}] \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & q_1(z_{s,1}) & \cdots & q_{\mu-2}(z_{s,1}) & f(z_{s,1}) \\ 0 & q_1[z_{s,1:2}] & \cdots & q_{\mu-2}[z_{s,1:2}] & f[z_{s,1:2}] \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & q_1[z_{s,1:\mu_s}] & \cdots & q_{\mu-2}[z_{s,1:\mu_s}] & f[z_{s,1:\mu_s}] \end{pmatrix}}{\prod_{\substack{1 \leq k < l \leq s \\ 1 \leq i \leq \mu_l, 1 \leq j \leq \mu_k}} (z_{l,i} - z_{k,j})}.$$

By using (2.5), we have that the numerator of the right-hand side of the above formula tends to  $\det(B_1^T, \dots, B_s^T)$  as  $z_{i,1:\mu_i} \rightarrow z_i^{(\mu_i)}$  for  $i = 1, \dots, s$ , where  $B_i$  ( $i = 1, \dots, s$ ) are given by (2.9). In particular, the denominator tends to  $\det(A_1^T, \dots, A_s^T)$ , where  $A_i$  are given by (2.8). Moreover, the value of  $\det(A_1^T, \dots, A_s^T)$  is

$$\prod_{\substack{1 \leq k < l \leq s \\ 1 \leq i \leq \mu_l, 1 \leq j \leq \mu_k}} \lim_{z_{l,i} \rightarrow z_{l,k,j} \rightarrow z_k} (z_{l,i} - z_{k,j}) = \prod_{1 \leq k < l \leq s} (z_l - z_k)^{\mu_l \mu_k}.$$

Hence

$$\begin{aligned} f[z_1^{(\mu_1)}, z_2^{(\mu_2)}, \dots, z_s^{(\mu_s)}] &= \lim_{z_{i,1:\mu_i} \rightarrow z_i^{(\mu_i)} (i=1,\dots,s)} f[z_{1,1:\mu_1}, \dots, z_{s,1:\mu_s}] \\ &= \frac{\det(B_1^T, \dots, B_s^T)}{\det(A_1^T, \dots, A_s^T)}. \quad \square \end{aligned}$$

I like to note that after the paper has been accepted for publication, Verde-Star kindly send me an old paper on divided difference [12], (4.1) of which gave a formula for divided difference with repeated nodes.

The following lemma gives a recursive formula for divided difference for polynomials, which is useful in proving Theorem 7 (for the entries of the inverse of band triangular Toeplitz matrices) in the next section.

**Lemma 4.** Suppose  $q_m(z) = z^m$  and  $z_1, \dots, z_s, z_{s+1} \in \mathbb{C}$  are different pairwise, then

$$q_{m+1}[z_{1:s+1}] = q_m[z_{1:s}] + z_{s+1}q_m[z_{1:s+1}]. \quad (2.10)$$

**Proof.** By using (2.1), we have

$$\begin{aligned} q_{m+1}[z_{1:s+1}] &= \sum_{j=1}^{s+1} \frac{z_j^{m+1}}{\prod_{r=1, r \neq j}^{s+1} (z_j - z_r)} \\ &= \sum_{j=1}^s \frac{z_j^m (z_j - z_{s+1} + z_{s+1})}{\prod_{r=1, r \neq j}^s (z_j - z_r) \cdot (z_j - z_{s+1})} + \frac{z_{s+1}^{m+1}}{\prod_{r=1}^s (z_{s+1} - z_r)} \\ &= \sum_{j=1}^s \frac{z_j^m}{\prod_{r=1, r \neq j}^s (z_j - z_r)} + \sum_{j=1}^s \frac{z_j^m z_{s+1}}{\prod_{r=1, r \neq j}^s (z_j - z_r)} + \frac{z_{s+1}^{m+1}}{\prod_{r=1}^s (z_{s+1} - z_r)} \end{aligned}$$



$$\begin{aligned}
&= q_m[z_{1:s}] + z_{s+1} \sum_{j=1}^s \frac{z_j^m}{\prod_{r=1, r \neq j}^s (z_j - z_r)} \\
&= q_m[z_{1:s}] + z_{s+1} q_m[z_{1:s+1}]. \quad \square
\end{aligned}$$

### 3. Inverse of band triangular Toeplitz matrices

Let  $p(\theta)$  be the trigonometric polynomial

$$p(\theta) = \prod_{j=1}^s (1 - z_j e^{i\theta})^{\mu_j}, \quad -\pi \leq \theta < \pi, \quad (3.1)$$

where  $z_1, \dots, z_s \in \mathbb{C}$  are different pairwise and  $\mu_j$  ( $j = 1, \dots, s$ ) are positive integers. In this section, we deduce an explicit formula for the entries of  $\mathcal{U}_n[p(\theta)]$ , i.e., the inverse of the band lower triangular Toeplitz matrix  $\mathcal{T}_n[p(\theta)]$  by using the results obtained in the previous section.

**Lemma 5.** *Here are some properties of triangular Toeplitz matrices.*

- (i) *Let  $T_1$  and  $T_2$  be two lower (upper) triangular Toeplitz matrices. Then  $T_1 T_2$  is also a lower (upper) triangular Toeplitz matrix, and*  

$$T_1 T_2 = T_2 T_1.$$
- (ii) *If  $T$  is an invertible lower (upper) triangular Toeplitz matrix, then  $T^{-1}$  is also a lower (upper) triangular Toeplitz matrix. Therefore, to obtain  $T^{-1}$ , we only require to compute the entries of its first column (row).*
- (iii) *Let  $f(\theta) = \sum_{j=0}^{\infty} t_j e^{ij\theta}$  and  $g(\theta) = \sum_{j=0}^{\infty} u_j e^{ij\theta}$ . Then*  

$$\mathcal{T}_n[fg] = \mathcal{T}_n[f] \mathcal{T}_n[g].$$
- (iv) *Let  $f$  and  $g$  be the same as in (iii). If  $t_0 \neq 0$  and  $u_0 \neq 0$ , then  $\mathcal{T}_n[f]$ ,  $\mathcal{T}_n[g]$ , and  $\mathcal{T}_n[fg]$  are invertible, moreover*  

$$\mathcal{U}_n[fg] = \mathcal{U}_n[f] \mathcal{U}_n[g].$$

**Proof.** The results can be verified directly.  $\square$

We first derive a formula for  $[\mathcal{U}_n[p]]_{j,1}$  ( $j = 1, \dots, n$ ) for the case when  $\mu_1 = \mu_2 = \dots = \mu_s = 1$ .

**Lemma 6.** *The first column of  $\mathcal{U}_n[1 - z_0 e^{i\theta}]$  is given by  $(1, z_0, z_0^2, \dots, z_0^{n-1})^T$ .*

**Proof.** Notice that  $\mathcal{T}_n[1 - z_0 e^{i\theta}]$  is the lower triangular Toeplitz matrix with the first column given by  $(1, -z_0, 0, \dots, 0)^T$ , we can verify the result directly.  $\square$

**Theorem 7.** *Let  $p(\theta)$  be the trigonometric polynomial defined in (3.1) with  $\mu_1 = \mu_2 = \dots = \mu_s = 1$  ( $s \geq 2$ ). Then the first column of  $\mathcal{U}_n[p(\theta)]$  is given by*

$$[\mathcal{U}_n[p(\theta)]]_{j,1} = q_{j+s-2}[z_{1:s}], \quad j = 1, 2, \dots, n, \quad (3.2)$$

where  $q_m(z) = z^m$  and  $f[z_{1:s}] = f[z_1, \dots, z_s]$  denotes the divided difference of order  $(s-1)$  for  $f(z)$ . In particular,  $[\mathcal{U}_n[p(\theta)]]_{1,1} = q_{s-1}[z_{1:s}] = 1$  (cf. (2.7)).

**Proof.** We prove the theorem by induction. Notice that the  $j$ th row of  $\mathcal{U}_n[1 - z_1 e^{i\theta}]$  and the first column of  $\mathcal{U}_n[1 - z_2 e^{i\theta}]$  are given by  $(z_1^{j-1}, z_1^{j-2}, \dots, 1, 0, \dots, 0)$  and  $(1, z_2, \dots, z_2^{n-1})^T$  respectively, by using property (iv) of Lemma 5 we have

$$\begin{aligned} & [\mathcal{U}_n[(1 - z_1 e^{i\theta})(1 - z_2 e^{i\theta})]]_{j,1} \\ &= z_1^{j-1} + z_1^{j-2} z_2 + \dots + z_1 z_2^{j-2} + z_2^{j-1} \\ &= (z_1^j - z_2^j) / (z_1 - z_2) = q_j[z_1, z_2] = q_{j+2-2}[z_1, z_2]. \end{aligned}$$

It follows that (3.2) is correct for  $s = 2$ .

Suppose (3.2) holds for  $s$ , we prove that it also holds for  $s + 1$  in the following. Obviously,

$$\left[ \mathcal{U}_n \left[ \prod_{j=1}^{s+1} (1 - z_j e^{i\theta}) \right] \right]_{1,1} = 1 = q_s[z_{1:s+1}].$$

By induction, the second row of  $\mathcal{U}_n[\prod_{j=1}^s (1 - z_j e^{i\theta})]$  is given by  $(q_s[z_{1:s}], 1, 0, \dots, 0)$ . Since the first column of  $\mathcal{U}_n[1 - z_{s+1} e^{i\theta}]$  is given by  $(1, z_{s+1}, \dots, z_{s+1}^{n-1})^T$ , we have

$$\begin{aligned} \left[ \mathcal{U}_n \left[ \prod_{j=1}^{s+1} (1 - z_j e^{i\theta}) \right] \right]_{2,1} &= q_s[z_{1:s}] + z_{s+1} \\ &= q_s[z_{1:s}] + z_{s+1} q_s[z_{1:s+1}] \\ &= q_{s+1}[z_{1:s+1}] \quad (\text{cf. (2.10)}). \end{aligned}$$

Similarly,

$$\begin{aligned} \left[ \mathcal{U}_n \left[ \prod_{j=1}^{s+1} (1 - z_j e^{i\theta}) \right] \right]_{3,1} &= q_{3+s-2}[z_{1:s}] + q_{3+s-3}[z_{1:s}] z_{s+1} + z_{s+1}^2 \\ &= q_{s+1}[z_{1:s}] + z_{s+1} (q_s[z_{1:s}] + z_{s+1}) \\ &= q_{s+1}[z_{1:s}] + z_{s+1} \left[ \mathcal{U}_n \left[ \prod_{j=1}^{s+1} (1 - z_j e^{i\theta}) \right] \right]_{2,1} \\ &= q_{s+1}[z_{1:s}] + z_{s+1} q_{s+1}[z_{1:s+1}] \\ &= q_{s+2}[z_{1:s+1}] = q_{3+(s+1)-2}[z_{1:s+1}]. \end{aligned}$$

By induction, the  $j$ th row of  $\mathcal{U}_n[\prod_{j=1}^s (1 - z_j e^{i\theta})]$  is given by

$$(q_{j+s-2}[z_{1:s}], q_{j+s-3}[z_{1:s}], \dots, 1, 0, \dots, 0)$$

and

$$\left[ \mathcal{U}_n \left[ \prod_{j=1}^{s+1} (1 - z_j e^{i\theta}) \right] \right]_{i,1} = q_{i+(s+1)-2}[z_{1:s+1}] = q_{i+s-1}[z_{1:s+1}], \quad i = 1, \dots, j-1.$$

Therefore

$$\begin{aligned}
& \left[ \mathcal{U}_n \left[ \prod_{j=1}^{s+1} (1 - z_j e^{i\theta}) \right] \right]_{j,1} \\
&= \left[ \mathcal{U}_n \left[ \prod_{j=1}^s (1 - z_j e^{i\theta}) \right] \mathcal{U}_n [1 - z_{s+1} e^{i\theta}] \right]_{j,1} \\
&= q_{j+s-2}[z_{1:s}] + q_{j+s-3}[z_{1:s}]z_{s+1} + \cdots + q_s[z_{1:s}]z_{s+1}^{j-2} + z_{s+1}^{j-1} \\
&= q_{j+s-2}[z_{1:s}] + z_{s+1} \left( q_{(j-1)+s-2}[z_{1:s}] + \cdots + q_s[z_{1:s}]z_{s+1}^{(j-1)-2} + z_{s+1}^{(j-1)-1} \right) \\
&= q_{j+s-2}[z_{1:s}] + z_{s+1} \left[ \mathcal{U}_n \left[ \prod_{j=1}^{s+1} (1 - z_j e^{i\theta}) \right] \right]_{j-1,1} \\
&= q_{j+s-2}[z_{1:s}] + z_{s+1} q_{j+s-2}[z_{1:s+1}] \\
&= q_{j+s-1}[z_{1:s+1}] = q_{j+(s+1)-2}[z_{1:s+1}].
\end{aligned}$$

Hence the formula (3.2) is correct for  $s + 1$ .  $\square$

By using the above theorem and Theorem 3, we have the following theorem about the entries of  $\mathcal{U}_n \left[ \prod_{i=1}^s (1 - z_i e^{i\theta})^{\mu_i} \right]$ .

**Theorem 8.** Let  $q_m(z) = z^m$  and

$$p(\theta) = \prod_{i=1}^s (1 - z_i e^{i\theta})^{\mu_i},$$

where  $z_1, \dots, z_s \in \mathbb{C}$  ( $s \geq 2$ ) are different pairwise and  $\mu_i$  are positive integers. Then the entries of the first column of  $\mathcal{U}_n[p(\theta)]$  are given by

$$\begin{aligned}
[\mathcal{U}_n[p(\theta)]]_{j,1} &= q_{j+\mu-2}[z_1^{(\mu_1)}, z_2^{(\mu_2)}, \dots, z_s^{(\mu_s)}] \\
&= \frac{\det(B_1^T, \dots, B_s^T)}{\prod_{1 \leq i < j \leq s} (z_j - z_i)^{\mu_i \mu_j}}, \quad j = 1, 2, \dots, n,
\end{aligned} \tag{3.3}$$

where  $\mu = \sum_{i=1}^s \mu_i$  and  $B_i$  ( $i = 1, \dots, s$ ) are given by (2.9) with  $f(z) = q_{j+\mu-2}(z) = z^{j+\mu-2}$ .

From (2.9) we see that  $(B_1^T, \dots, B_s^T)$  is a  $\mu \times \mu$  matrix and its first  $\mu - 1$  rows are determined by  $z_1, \dots, z_s$  and  $\mu_1, \dots, \mu_s$ . Let  $(B_1^T, \dots, B_s^T)$  be denoted by  $[b_{ij}]_{i,j=1}^{\mu}$ . Notice that

$$\det([b_{ij}]_{i=1:\mu}^{j=1:\mu}) = \sum_{k=1}^{\mu} (-1)^{\mu+k} b_{\mu,k} \det([b_{ij}]_{i=1:\mu-1}^{j=1:n, j \neq k}),$$

from (3.3) we get

$$\begin{aligned}
[\mathcal{U}_n[p]]_{j,1} &= \sum_{i=1}^s \sum_{l=0}^{\mu_i-1} \alpha_{i,l} q_{j+\mu-2}^{(l)}(z_i) \\
&= \sum_{i=1}^s \alpha_{i,0} z_i^{j+\mu-2} + \sum_{i=1}^s \sum_{l=1}^{\mu_i-1} \left( \alpha_{i,l} \prod_{r=1}^l (j + \mu - r - 1) \right) z_i^{j+\mu-l-2},
\end{aligned} \tag{3.4}$$

where  $\alpha_{i,l}$ ,  $i = 1, \dots, s$ ,  $l = 0, 1, \dots, \mu_i - 1$  are determined by  $z_1, z_2, \dots, z_s$  and  $\mu_1, \dots, \mu_s$ .

We observe from (3.4) that

1. We can compute the entries  $[\mathcal{U}_n[p]]_{j,1}$  ( $j = 1, \dots, n$ ) in parallel in  $O(\mu)$  operations by first computing the coefficients  $\{\alpha_{i,l} : i = 1, \dots, s, l = 0, 1, \dots, \mu_i - 1\}$  and then applying formula (3.4). In fact, we can obtain the first column of  $\mathcal{U}_n[p]$  in  $O(n\mu)$  operations by direct triangular solver (in the order  $[\mathcal{U}_n[p]]_{1,1} \rightarrow [\mathcal{U}_n[p]]_{2,1} \rightarrow \dots \rightarrow [\mathcal{U}_n[p]]_{n,1}$ ).
2. If  $|z_i| < 1$  for  $i = 1, \dots, s$ , then the matrix  $\mathcal{T}_n[p]$  is well-conditioned; if there exists an  $i_0$  such that  $|z_{i_0}| > 1$ , then the matrix  $\mathcal{T}_n[p]$  is severely ill-conditioned; if  $|z_i| = 1$  for  $i = 1, \dots, s$ , i.e., the case when  $p(\theta)$  is defined by (1.1), then

$$[\mathcal{U}_n[p]]_{j,1} = O(j^{\mu_{\max}-1}), \quad (3.5)$$

where  $\mu_{\max} = \max\{\mu_1, \mu_2, \dots, \mu_s\}$ . It follows that the condition number of  $\mathcal{T}_n[p]$  is  $O(n^{\mu_{\max}})$ . We will use (3.5) to estimate the condition number of  $\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}]$  in the next section.

#### 4. The condition number of $\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}]$

In this section, we first prove the conjecture proposed in [8], i.e., the minimal eigenvalue of the matrix  $\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}]$  is well separated from the origin, where

$$p(\theta) = \prod_{i=1}^s (1 - e^{i(\theta-\theta_i)})^{\mu_i} \quad (4.1)$$

and

$$f(\theta) = h(\theta) \prod_{i=1}^s (1 - \cos(\theta - \theta_i))^{\mu_i} \quad (4.2)$$

with  $h \in \mathcal{C}_{2\pi}$  with  $h(\theta) > 0$ . Here  $\theta_i$ ,  $i = 1, \dots, s$ , are different numbers in  $[-\pi, \pi)$  and  $\mu_1, \dots, \mu_s$  are positive integers. We then estimate the condition number of  $\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}]$  by using (3.5) and the above conjecture.

We first prove that the minimal eigenvalue of  $\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}]$  is not less than  $2^{-\mu}h_{\min}$ , where  $\mu = \sum_{i=1}^s \mu_i$  and  $h_{\min} = \min_{-\pi \leq \theta < \pi} h(\theta)$ .

**Lemma 9.** *The minimal eigenvalue of  $\mathcal{U}_n[p]\mathcal{T}_n[p\bar{p}]\mathcal{U}_n[\bar{p}]$  is not less than 1, where  $p(\theta)$  is defined by (4.1).*

**Proof.** Suppose  $p(\theta) = a_0 + a_1 e^{i\theta} + \dots + a_\mu e^{i\mu\theta}$ , we have

$$\begin{aligned} p(\theta)\overline{p(\theta)} &= \sum_{j=0}^{\mu} |a_j|^2 + \left( \sum_{j=0}^{\mu-1} \bar{a}_j a_{j+1} e^{i\theta} + \sum_{j=0}^{\mu-1} a_j \bar{a}_{j+1} e^{-i\theta} \right) \\ &\quad + \dots + (\bar{a}_0 a_\mu e^{i\mu\theta} + a_0 \bar{a}_\mu e^{-i\mu\theta}). \end{aligned}$$

It can be easily checked that

$$\mathcal{T}_n[p\bar{p}] - \mathcal{T}_n[p]\mathcal{T}_n[\bar{p}] = B_n B_n^*,$$

where

$$B_n = \begin{pmatrix} B_\mu & O_{\mu \times (n-\mu)} \\ O_{(n-\mu) \times \mu} & O_{(n-\mu) \times (n-\mu)} \end{pmatrix} \quad \text{with } B_\mu = \begin{pmatrix} \bar{a}_1 & \bar{a}_2 & \dots & \bar{a}_\mu \\ & \bar{a}_1 & \ddots & \vdots \\ & & \ddots & \vdots \\ & & & \bar{a}_1 \end{pmatrix}. \quad (4.3)$$

Thus

$$\mathcal{U}_n[p] \mathcal{T}_n[p\bar{p}] \mathcal{U}_n[\bar{p}] - I = (\mathcal{U}_n[p] B_n) (\mathcal{U}_n[p] B_n)^* \quad (4.4)$$

is Hermitian and semi-positive and the result follows.  $\square$

The following theorem gives a proof of the conjecture proposed in [8] (end of page 227).

**Theorem 10.** Let  $p(\theta)$  and  $f(\theta)$  be defined by (4.1) and (4.2) respectively. Then the minimal eigenvalue of  $\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}]$  is not less than  $2^{-\mu} h_{\min}$ .

**Proof.** From Lemma 9, we have for  $\mathbf{x} \neq 0$ ,

$$\frac{\mathbf{x}^* \mathcal{U}_n[p] \mathcal{T}_n[p\bar{p}] \mathcal{U}_n[\bar{p}] \mathbf{x}}{\mathbf{x}^* \mathbf{x}} \geq 1.$$

Therefore, for  $\mathbf{x} \neq 0$ ,

$$\begin{aligned} & \frac{\mathbf{x}^* \mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}] \mathbf{x}}{\mathbf{x}^* \mathbf{x}} \\ &= \frac{\mathbf{x}^* \mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}] \mathbf{x}}{\mathbf{x}^* \mathcal{U}_n[p] \mathcal{T}_n[p\bar{p}] \mathcal{U}_n[\bar{p}] \mathbf{x}} \cdot \frac{\mathbf{x}^* \mathcal{U}_n[p] \mathcal{T}_n[p\bar{p}] \mathcal{U}_n[\bar{p}] \mathbf{x}}{\mathbf{x}^* \mathbf{x}} \\ &\geq \frac{\mathbf{x}^* \mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}] \mathbf{x}}{\mathbf{x}^* \mathcal{U}_n[p] \mathcal{T}_n[p\bar{p}] \mathcal{U}_n[\bar{p}] \mathbf{x}} \\ &= \frac{\mathbf{y}^* \mathcal{T}_n[f] \mathbf{y}}{\mathbf{y}^* \mathcal{T}_n[p\bar{p}] \mathbf{y}} \quad (\mathbf{y} = \mathcal{U}_n[\bar{p}] \mathbf{x} \neq 0). \end{aligned}$$

Note that  $f(\theta)/[p(\theta)\overline{p(\theta)}] \geq 2^{-\mu} h_{\min}$ , we have

$$\begin{aligned} \mathbf{y}^* \mathcal{T}_n[f] \mathbf{y} &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{j=1}^n y_j e^{-ij\theta} \right|^2 f(\theta) d\theta \\ &\geq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \sum_{j=1}^n y_j e^{-ij\theta} \right|^2 2^{-\mu} h_{\min} p(\theta) \overline{p(\theta)} d\theta \\ &= 2^{-\mu} h_{\min} \mathbf{y}^* \mathcal{T}_n[p\bar{p}] \mathbf{y}. \end{aligned}$$

Hence

$$\min_{\mathbf{x} \neq 0} \frac{\mathbf{x}^* \mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}] \mathbf{x}}{\mathbf{x}^* \mathbf{x}} \geq \min_{\mathbf{y} \neq 0} \frac{\mathbf{y}^* \mathcal{T}_n[f] \mathbf{y}}{\mathbf{y}^* \mathcal{T}_n[p\bar{p}] \mathbf{y}} \geq 2^{-\mu} h_{\min},$$

that is, the minimal eigenvalue of  $\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}]$  is not less than  $2^{-\mu} h_{\min}$ .  $\square$

The following is concerned with the condition number of  $\mathcal{U}_n[p] \mathcal{T}_n[f] \mathcal{U}_n[\bar{p}]$ .

**Lemma 11.** Let  $p(\theta)$  be defined by (4.1) and  $\mu_{\max} = \max\{\mu_1, \dots, \mu_s\}$ . Then the condition number of  $\mathcal{U}_n[p]\mathcal{T}_n[p\bar{p}]\mathcal{U}_n[\bar{p}]$  is bounded by  $O(n^{2\mu_{\max}-1})$ .

**Proof.** Since the minimal eigenvalue of  $\mathcal{U}_n[p]\mathcal{T}_n[p\bar{p}]\mathcal{U}_n[\bar{p}]$  is not less than 1, we only require to prove that the maximal eigenvalue, i.e. the 2-norm of  $\mathcal{U}_n[p]\mathcal{T}_n[p\bar{p}]\mathcal{U}_n[\bar{p}]$  is bounded by  $O(n^{2\mu_{\max}-1})$ . By using (4.4), we only require to prove that  $\|\mathcal{U}_n[p]B_n\|_2^2$  is bounded by  $O(n^{2\mu_{\max}-1})$ .

From (3.5), we see that

$$[\mathcal{U}_n[p]]_{j,k} = O((j-k+1)^{\mu_{\max}-1}), \quad 1 \leq k \leq j \leq n.$$

Therefore, by using (4.3) we have that there exists  $c_1 > 0$  such that

$$|[\mathcal{U}_n[p]B_n]_{j,k}| = \begin{cases} \leq c_1 j^{\mu_{\max}-1}, & k \leq \mu, \\ 0, & k > \mu. \end{cases}$$

Thus,

$$\|\mathcal{U}_n[p]B_n\|_{\infty} \leq \mu c_1 n^{\mu_{\max}-1}$$

and

$$\|\mathcal{U}_n[p]B_n\|_1 \leq c_1(1 + 2^{\mu_{\max}-1} + \dots + n^{\mu_{\max}-1}) \leq c_2 n^{\mu_{\max}}$$

for some constant  $c_2 > 0$ . It follows that

$$\begin{aligned} \|\mathcal{U}_n[p]B_n\|_2^2 &\leq \|\mathcal{U}_n[p]B_n\|_{\infty} \|\mathcal{U}_n[p]B_n\|_1 \\ &\leq \mu c_1 n^{\mu_{\max}-1} c_2 n^{\mu_{\max}} = \mu c_1 c_2 n^{2\mu_{\max}-1}. \quad \square \end{aligned}$$

**Theorem 12.** Let  $p(\theta)$  and  $f(\theta)$  be defined by (4.1) and (4.2) respectively, and  $\mu_{\max} = \max\{\mu_1, \dots, \mu_s\}$ . Then the condition number of  $\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}]$  is bounded by  $O(n^{2\mu_{\max}-1})$ .

**Proof.** By using Lemma 11 and Theorem 10, the theorem can be proved similarly as we prove Theorem 10.  $\square$

Since  $f/(p\bar{p}) = f/(2^\mu w) > 0$ , the matrix  $\mathcal{T}_n[f/(2^\mu w)]$  is Hermitian and positive definite. Thus we can choose well-conditioned positive definite matrix  $Q_n$  in Theorem 1 such that (1.2) holds. For such  $Q_n$ , the matrix  $Q_n\{\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}]\}$  satisfies: (1) the spectrum is clustered around 1; (2) the minimal eigenvalue is well separated from the origin; and (3) the condition number is bounded by  $O(n^{2\mu_{\max}-1})$ . Therefore it follows that the PCG method converges very fast when applied to solve the preconditioned system  $Q_n\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}](\mathcal{T}_n[\bar{p}]\mathbf{x}) = Q_n\mathcal{U}_n[p]\mathbf{b}$ .

## 5. Concluding remarks

In this paper, we derive an explicit formula for the entries of the inverse of band lower triangular Toeplitz matrices by means of divided difference with repeated nodes. We then prove the conjecture proposed in [8] that the minimal eigenvalue of  $\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}]$  is well separated from the origin. Finally, by using the explicit formula for  $[\mathcal{U}_n[p]]_{j,1}$  ( $j = 1, \dots, n$ ) and the conjecture, we show that the condition number of  $\mathcal{U}_n[p]\mathcal{T}_n[f]\mathcal{U}_n[\bar{p}]$  is bounded by  $O(n^{\nu-1})$ , where  $n$  is the size of the system and  $\nu$  is the maximal order of the zeros of the generating function  $f$ .

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